

Cross-intersecting sub-families of hereditary families

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Abstract

Families $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ of sets are said to be *cross-intersecting* if for any i and j in $\{1, 2, \dots, k\}$ with $i \neq j$, any set in \mathcal{A}_i intersects any set in \mathcal{A}_j . For a finite set X , let 2^X denote the *power set of X* (the family of all subsets of X). A family \mathcal{H} is said to be *hereditary* if all subsets of any set in \mathcal{H} are in \mathcal{H} ; so \mathcal{H} is hereditary if and only if it is a union of power sets. We conjecture that for any non-empty hereditary sub-family $\mathcal{H} \neq \{\emptyset\}$ of 2^X and any $k \geq |X| + 1$, both the sum and product of sizes of k cross-intersecting sub-families $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ (not necessarily distinct or non-empty) of \mathcal{H} are maxima if $\mathcal{A}_1 = \mathcal{A}_2 = \dots = \mathcal{A}_k = \mathcal{S}$ for some largest *star \mathcal{S} of \mathcal{H}* (a sub-family of \mathcal{H} whose sets have a common element). We prove this for the case when \mathcal{H} is *compressed with respect to an element x of X* , and for this purpose we establish new properties of the usual *compression operation*. For the product, we actually conjecture that the configuration $\mathcal{A}_1 = \mathcal{A}_2 = \dots = \mathcal{A}_k = \mathcal{S}$ is optimal for any hereditary \mathcal{H} and any $k \geq 2$, and we prove this for a special case too.

1 Basic definitions and notation

Unless otherwise stated, we shall use small letters such as x to denote elements of a set or non-negative integers or functions, capital letters such as X to denote sets, and calligraphic letters such as \mathcal{F} to denote *families* (i.e. sets whose elements are sets themselves). It is to be assumed that sets and families are *finite*. We call a set A an *r -element set*, or simply an *r -set*, if its size $|A|$ is r (i.e. if it contains exactly r elements).

For any integer $n \geq 1$, the set $\{1, \dots, n\}$ of the first n positive integers is denoted by $[n]$. For a set X , the *power set of X* (i.e. $\{A: A \subseteq X\}$) is denoted by 2^X , and the family of all r -element subsets of X is denoted by $\binom{X}{r}$.

A family \mathcal{H} is said to be a *hereditary family* (also called an *ideal* or a *downset*) if all the subsets of any set in \mathcal{H} are in \mathcal{H} . Clearly a family is hereditary if and only if it is a union of power sets. A *base of \mathcal{H}* is a set in \mathcal{H} that is not a subset of any other set in \mathcal{H} .

So a hereditary family is the union of power sets of its bases. An interesting example of a hereditary family is the family of all independent sets of a graph or matroid.

We will denote the union of all sets in a family \mathcal{F} by $U(\mathcal{F})$. If x is an element of a set X , then we denote the family of those sets in \mathcal{F} which contain x by $\mathcal{F}\langle x \rangle$, and we call $\mathcal{F}\langle x \rangle$ a *star of \mathcal{F}* . So $\mathcal{F}\langle x \rangle$ is the empty set \emptyset if and only if x is not in $U(\mathcal{F})$.

A family \mathcal{A} is said to be *intersecting* if any two sets in \mathcal{A} intersect (i.e. contain at least one common element). We call a family \mathcal{A} *centred* if the sets in \mathcal{A} have a common element x (i.e. $\mathcal{A} = \mathcal{A}\langle x \rangle$). So a centred family is intersecting, and a non-empty star of a family \mathcal{F} is centred. The simplest example of a non-centred intersecting family is $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ (i.e. $\binom{[3]}{2}$).

Families $\mathcal{A}_1, \dots, \mathcal{A}_k$ are said to be *cross-intersecting* if for any i and j in $[k]$ with $i \neq j$, any set in \mathcal{A}_i intersects any set in \mathcal{A}_j .

If $U(\mathcal{F})$ has an element x such that $\mathcal{F}\langle x \rangle$ is a largest intersecting sub-family of \mathcal{F} (i.e. no intersecting sub-family of \mathcal{F} has more sets than $\mathcal{F}\langle x \rangle$), then we say that \mathcal{F} has the *star property at x* . We simply say that \mathcal{F} has the *star property* if either $U(\mathcal{F}) = \emptyset$ or \mathcal{F} has the star property at some element of $U(\mathcal{F})$.

If $U(\mathcal{F})$ has an element x such that $(F \setminus \{y\}) \cup \{x\} \in \mathcal{F}$ whenever $y \in F \in \mathcal{F}$ and $x \notin F$, then \mathcal{F} is said to be *compressed with respect to x* . For example, this is the case when \mathcal{F} is the family of all independent sets of a graph that has an isolated vertex x .

A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be *left-compressed* if $(F \setminus \{j\}) \cup \{i\} \in \mathcal{F}$ whenever $1 \leq i < j \in F \in \mathcal{F}$ and $i \notin F$.

2 Intersecting sub-families of hereditary families

The following is a famous longstanding open conjecture in extremal set theory due to Chvátal (see [7] for a more general conjecture).

Conjecture 2.1 ([10]) *If \mathcal{H} is a hereditary family, then \mathcal{H} has the star property.*

This conjecture was verified for the case when \mathcal{H} is left-compressed by Chvátal [11] himself. Snevily [17] took this result (together with results in [16, 18]) a significant step forward by verifying Conjecture 2.1 for the case when \mathcal{H} is compressed with respect to an element x of $U(\mathcal{H})$.

Theorem 2.2 ([17]) *If a hereditary family \mathcal{H} is compressed with respect to an element x of $U(\mathcal{H})$, then \mathcal{H} has the star property at x .*

A generalisation is proved in [7] by means of an alternative self-contained argument.

Snevily's proof of Theorem 2.2 makes use of the following interesting result of Berge [2] (a proof of which is also provided in [1, Chapter 6]).

Theorem 2.3 ([2]) *If \mathcal{H} is a hereditary family, then \mathcal{H} is a disjoint union of pairs of disjoint sets, together with \emptyset if $|\mathcal{H}|$ is odd.*

This result was also motivated by Conjecture 2.1 as it has the following immediate consequence.

Corollary 2.4 *If \mathcal{A} is an intersecting sub-family of a hereditary family \mathcal{H} , then*

$$|\mathcal{A}| \leq \frac{1}{2}|\mathcal{H}|.$$

Proof. For any pair of disjoint sets, at most only one set can be in an intersecting family \mathcal{A} . By Theorem 2.3, the result follows. \square

A special case of Theorem 2.2 is a result of Schönheim [16] which says that Conjecture 2.1 is true if the bases of \mathcal{H} have a common element, and this follows immediately from Corollary 2.4 and the following fact.

Proposition 2.5 *If the bases of a hereditary family \mathcal{H} have a common element x , then*

$$|\mathcal{H}\langle x \rangle| = \frac{1}{2}|\mathcal{H}|.$$

Proof. By induction on $|U(\mathcal{H})|$. We have $x \in U(\mathcal{H})$. Let $\mathcal{A} = \mathcal{H}\langle x \rangle$. If $|U(\mathcal{H})| = 1$, then $\mathcal{H} = \{\emptyset, \{x\}\}$, $\mathcal{A} = \{\{x\}\}$ and hence $|\mathcal{A}| = 1 = \frac{1}{2}|\mathcal{H}|$. Now suppose $|U(\mathcal{H})| \geq 2$. Then $U(\mathcal{H})$ has an element $y \neq x$. Let $\mathcal{I} = \{H \setminus \{y\} : H \in \mathcal{H}\langle y \rangle\}$ and $\mathcal{J} = \{H \in \mathcal{H} : y \notin H\}$. Similarly, let $\mathcal{B} = \{A \setminus \{y\} : A \in \mathcal{A}\langle y \rangle\}$ and $\mathcal{C} = \{A \in \mathcal{A} : y \notin A\}$. Note that \mathcal{I} and \mathcal{J} are hereditary, the bases of \mathcal{I} and \mathcal{J} contain x , $|U(\mathcal{I})| \leq |U(\mathcal{H})| - 1$, $|U(\mathcal{J})| \leq |U(\mathcal{H})| - 1$, $\mathcal{B} = \mathcal{I}\langle x \rangle$ and $\mathcal{C} = \mathcal{J}\langle x \rangle$. By the inductive hypothesis, $|\mathcal{B}| = \frac{1}{2}|\mathcal{I}|$ and $|\mathcal{C}| = \frac{1}{2}|\mathcal{J}|$. Finally, $|\mathcal{A}| = |\mathcal{A}\langle y \rangle| + |\mathcal{C}| = |\mathcal{B}| + |\mathcal{C}| = \frac{1}{2}(|\mathcal{I}| + |\mathcal{J}|) = \frac{1}{2}(|\mathcal{H}\langle y \rangle| + |\mathcal{J}|) = \frac{1}{2}|\mathcal{H}|$. \square

Many other results and problems have been inspired by Conjecture 2.1 or are related to it; see [9, 15, 19].

3 Cross-intersecting sub-families of hereditary families

For intersecting sub-families of a given family \mathcal{F} , the natural question to ask is how large they can be. Conjecture 2.1 claims that when \mathcal{F} is hereditary, we need only check the stars of \mathcal{F} (of which there are $|U(\mathcal{F})|$). For cross-intersecting families, two natural parameters arise: the sum and the product of sizes of the cross-intersecting families (note that the product of sizes of k families $\mathcal{A}_1, \dots, \mathcal{A}_k$ is the number of k -tuples (A_1, \dots, A_k) such that $A_i \in \mathcal{A}_i$ for each $i \in [k]$). It is therefore natural to consider the problem of maximising the sum or the product of sizes of k cross-intersecting sub-families (not necessarily distinct or non-empty) of a given family \mathcal{F} (see [8]). We suggest a few conjectures for the case when \mathcal{F} is hereditary, and we prove that they are true in some important cases. Obviously, any family \mathcal{F} is a sub-family of 2^X with $X = U(\mathcal{F})$, and we may assume that $X = [n]$.

For the sum of sizes, we suggest the following.

Conjecture 3.1 *If $k \geq n + 1$ and $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-intersecting sub-families of a hereditary sub-family $\mathcal{H} \neq \{\emptyset\}$ of $2^{[n]}$, then the sum $\sum_{i=1}^k |\mathcal{A}_i|$ is maximum if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$ for some largest star \mathcal{S} of \mathcal{H} .*

We cannot remove the condition that $k \geq n + 1$. Indeed, consider $\mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$ and $2 \leq k < n + 1$. Let $\mathcal{S} = \{\{1\}\}$; so \mathcal{S} is a largest star of \mathcal{H} . Let $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$, and let $\mathcal{B}_1 = \mathcal{H}$ and $\mathcal{B}_2 = \dots = \mathcal{B}_k = \emptyset$. Then $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-intersecting, $\mathcal{B}_1, \dots, \mathcal{B}_k$ are cross-intersecting, and $\sum_{i=1}^k |\mathcal{A}_i| = k < n + 1 = \sum_{i=1}^k |\mathcal{B}_i|$. Also, we cannot remove the condition that $\mathcal{H} \neq \{\emptyset\}$. Indeed, suppose $\mathcal{H} = \{\emptyset\}$; so $\mathcal{S} = \emptyset$. Thus, if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$, $\mathcal{B}_1 = \mathcal{H}$ and $\mathcal{B}_2 = \dots = \mathcal{B}_k = \emptyset$, then $\sum_{i=1}^k |\mathcal{A}_i| = 0 < 1 = \sum_{i=1}^k |\mathcal{B}_i|$.

For the general case when we have any number of cross-intersecting families, we suggest the following stronger conjecture.

Conjecture 3.2 *Let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be cross-intersecting sub-families of a non-empty hereditary sub-family $\mathcal{H} \neq \{\emptyset\}$ of $2^{[n]}$, and let \mathcal{S} be a largest star of \mathcal{H} .*

- (i) *If $k \leq \frac{|\mathcal{H}|}{|\mathcal{S}|}$, then $\sum_{i=1}^k |\mathcal{A}_i|$ is maximum if $\mathcal{A}_1 = \mathcal{H}$ and $\mathcal{A}_2 = \dots = \mathcal{A}_k = \emptyset$.*
- (ii) *If $k \geq \frac{|\mathcal{H}|}{|\mathcal{S}|}$, then $\sum_{i=1}^k |\mathcal{A}_i|$ is maximum if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$.*

This conjecture is simply saying that at least one of the two simple configurations $\mathcal{A}_1 = \mathcal{H}$, $\mathcal{A}_2 = \dots = \mathcal{A}_k = \emptyset$ and $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$ gives a maximum sum of sizes. This strengthens Conjecture 3.1 because, since \mathcal{H} has a non-empty set (as $\mathcal{H} \neq \emptyset$ and $\mathcal{H} \neq \{\emptyset\}$), we have $\mathcal{S} \neq \emptyset$, $|\mathcal{H}| = |\{\emptyset\} \cup \bigcup_{i=1}^n \mathcal{H}\langle i \rangle| \leq 1 + n|\mathcal{S}| \leq (n + 1)|\mathcal{S}|$ and hence $\frac{|\mathcal{H}|}{|\mathcal{S}|} \leq n + 1$; that is, if (ii) is true, then Conjecture 3.1 follows.

For the product of sizes, we first present the following consequence of Conjecture 3.1.

Proposition 3.3 *Let $\mathcal{A}_1, \dots, \mathcal{A}_k, \mathcal{H}$ and \mathcal{S} be as in Conjecture 3.1. If Conjecture 3.1 is true, then the product $\prod_{i=1}^k |\mathcal{A}_i|$ is maximum if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$.*

This follows immediately from the following elementary result, known as the Arithmetic Mean-Geometric Mean (AM-GM) Inequality.

Lemma 3.4 (AM-GM Inequality) *If x_1, x_2, \dots, x_k are non-negative real numbers, then*

$$\left(\prod_{i=1}^k x_i \right)^{1/k} \leq \frac{1}{k} \sum_{i=1}^k x_i.$$

Indeed, suppose Conjecture 3.1 is true. Then $\sum_{i=1}^k |\mathcal{A}_i| \leq k|\mathcal{S}|$. Thus, by Lemma 3.4, $\left(\prod_{i=1}^k |\mathcal{A}_i| \right)^{1/k} \leq |\mathcal{S}|$ and hence Proposition 3.3.

However, we conjecture the following stronger statement about the maximum product.

Conjecture 3.5 *If $k \geq 2$ and $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-intersecting sub-families of a hereditary family \mathcal{H} , then $\prod_{i=1}^k |\mathcal{A}_i|$ is maximum if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$ for some largest star \mathcal{S} of \mathcal{H} .*

If the above conjecture is true for $k = 2$, then it is true for any $k \geq 2$. Indeed, it is not difficult to show that, in general, if $p \geq 2$ and \mathcal{L} is an intersecting sub-family of a family \mathcal{F} such that the product of sizes of p cross-intersecting sub-families of \mathcal{F} is a maximum when each of them is \mathcal{L} , then for any $k \geq p$, the product of sizes of k cross-intersecting sub-families of \mathcal{F} is also maximum when each of them is \mathcal{L} ; see [8].

Each of the above conjectures generalises Conjecture 2.1. Indeed, let \mathcal{A} be an intersecting sub-family of a hereditary family $\mathcal{H} \subseteq 2^{[n]}$ with $U(\mathcal{H}) \neq \emptyset$, and let \mathcal{S} be a largest star of \mathcal{H} . Let $k \geq n + 1$, and let $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{A}$. Then $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-intersecting. Thus, any of Conjectures 3.1, 3.2 and 3.5 claims that $|\mathcal{A}_i| \leq |\mathcal{S}|$ for each $i \in [k]$ (since $\mathcal{A}_1 = \dots = \mathcal{A}_k$), and hence $|\mathcal{A}| \leq |\mathcal{S}|$ as claimed by Conjecture 2.1.

All the above conjectures are true for the special case when $\mathcal{H} = 2^{[n]}$; more precisely, the following holds.

Theorem 3.6 ([8]) *For any $k \geq 2$, both the sum and product of sizes of k cross-intersecting sub-families $\mathcal{A}_1, \dots, \mathcal{A}_k$ of $2^{[n]}$ are maxima if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \{A \subseteq [n] : 1 \in A\}$.*

We generalise this result as follows.

Theorem 3.7 *If $\mathcal{A}_1, \dots, \mathcal{A}_k$ are cross-intersecting sub-families of a hereditary family \mathcal{H} , then*

$$\sum_{i=1}^k |\mathcal{A}_i| \leq k \frac{|\mathcal{H}|}{2} \quad \text{and} \quad \prod_{i=1}^k |\mathcal{A}_i| \leq \left(\frac{|\mathcal{H}|}{2} \right)^k.$$

Moreover, both bounds are attained if the bases of \mathcal{H} have a common element x and $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{H}\langle x \rangle$.

Proof. Theorem 2.3 tells us that there exists a partition $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \dots \cup \mathcal{H}_m$ of \mathcal{H} such that $m = \left\lceil \frac{|\mathcal{H}|}{2} \right\rceil$, $\mathcal{H}_i = \{H_{i,1}, H_{i,2}\}$ for some $H_{i,1}, H_{i,2} \in \mathcal{H}$ with $H_{i,1} \cap H_{i,2} = \emptyset$, $i = 1, \dots, m$, and if $|\mathcal{H}|$ is odd then $H_{m,1} = H_{m,2} = \emptyset$.

Let $\mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i$. By the cross-intersection condition, we clearly have $\mathcal{A}^* = \bigcup_{i=1}^k \mathcal{A}_i^*$ and $\mathcal{A}' = \bigcup_{i=1}^k \mathcal{A}_i'$. Suppose $\mathcal{A}_i' \cap \mathcal{A}_j' \neq \emptyset$ for some $i \neq j$. Let $A \in \mathcal{A}_i' \cap \mathcal{A}_j'$. Then there exists $A_i \in \mathcal{A}_i'$ such that $A \cap A_i = \emptyset$, but this is a contradiction because $A \in \mathcal{A}_j$. So $\mathcal{A}_i' \cap \mathcal{A}_j' = \emptyset$ for any $i \neq j$. Therefore $|\mathcal{A}'| = \sum_{i=1}^k |\mathcal{A}_i'|$.

Let $\mathcal{B} = \{H_{i,j} : i \in [m], j \in [2], H_{i,3-j} \in \mathcal{A}^*\}$. So $|\mathcal{B}| = |\mathcal{A}^*|$. For any $H_{i,j} \in \mathcal{B}$, $H_{i,j} \notin \mathcal{A}$ since $H_{i,3-j} \in \mathcal{A}^*$ and $H_{i,j} \cap H_{i,3-j} = \emptyset$. So \mathcal{A} and \mathcal{B} are disjoint sub-families of \mathcal{H} . Therefore,

$$2|\mathcal{A}^*| + |\mathcal{A}'| = |\mathcal{A}^*| + |\mathcal{B}| + |\mathcal{A}'| = |\mathcal{A}| + |\mathcal{B}| = |\mathcal{A} \cup \mathcal{B}| \leq |\mathcal{H}|$$

and hence, dividing throughout by 2, we get $|\mathcal{A}^*| + \frac{1}{2}|\mathcal{A}'| \leq \frac{1}{2}|\mathcal{H}|$. So we have

$$\sum_{i=1}^k |\mathcal{A}_i| = \sum_{i=1}^k |\mathcal{A}_i'| + \sum_{i=1}^k |\mathcal{A}_i^*| \leq |\mathcal{A}'| + k|\mathcal{A}^*| \leq k \left(|\mathcal{A}^*| + \frac{1}{2}|\mathcal{A}'| \right) \leq k \frac{|\mathcal{H}|}{2}$$

and hence, by Lemma 3.4,

$$\prod_{i=1}^k |\mathcal{A}_i| \leq \left(\frac{1}{k} \sum_{i=1}^k |\mathcal{A}_i| \right)^k \leq \left(\frac{|\mathcal{H}|}{2} \right)^k.$$

The second part of the theorem is an immediate consequence of Proposition 2.5. \square

Corollary 3.8 *Conjectures 3.1, 3.2 and 3.5 are true if the bases of \mathcal{H} have a common element.*

Proof. If the bases of \mathcal{H} have a common element x , then by Corollary 2.4 and Proposition 2.5, $\mathcal{H}\langle x \rangle$ is a largest star of \mathcal{H} of size $|\mathcal{H}|/2$. By Theorem 3.7, the result follows. \square

Corollary 3.9 *Conjecture 3.2 is true if $k = 2$.*

Proof. By Corollary 2.4, we have $|\mathcal{S}| \leq |\mathcal{H}|/2$ and hence $2 \leq \frac{|\mathcal{H}|}{|\mathcal{S}|}$. Now by Theorem 3.7, $|\mathcal{A}_1| + |\mathcal{A}_2| \leq |\mathcal{H}|$. Hence the result. \square

We now come to our main result, which verifies Conjectures 3.1, 3.2 and 3.5 for the case when $k \geq n + 1$ and \mathcal{H} is compressed with respect to an element of $[n]$. As remarked in Section 2, an important example of such a hereditary family is one whose bases have a common element. Other important examples include $\bigcup_{r=0}^m \binom{[n]}{r}$ for any $m \in \{0\} \cup [n]$ (for $m = n$ we get $2^{[n]}$).

Theorem 3.10 *Let \mathcal{H} be a hereditary sub-family of $2^{[n]}$ that is compressed with respect to an element x of $[n]$, and let $\mathcal{S} = \mathcal{H}\langle x \rangle$. Let $k \geq n + 1$, and let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be cross-intersecting sub-families of \mathcal{H} . Then*

$$\sum_{i=1}^k |\mathcal{A}_i| \leq k|\mathcal{S}| \quad \text{and} \quad \prod_{i=1}^k |\mathcal{A}_i| \leq |\mathcal{S}|^k,$$

and both bounds are attained if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{S}$. Moreover:

(a) $\sum_{i=1}^k |\mathcal{A}_i| = k|\mathcal{S}|$ if and only if either $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{L}$ for some largest intersecting sub-family \mathcal{L} of \mathcal{H} or $k = n + 1$ and for some $i \in [k]$, $\mathcal{A}_i = \mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$ and $\mathcal{A}_j = \emptyset$ for each $j \in [k] \setminus \{i\}$.

(b) $\prod_{i=1}^k |\mathcal{A}_i| = |\mathcal{S}|^k$ if and only if $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{L}$ for some largest intersecting sub-family \mathcal{L} of \mathcal{H} .

This generalises Theorem 2.2 in the same way that Conjectures 3.1, 3.2 and 3.5 generalise Conjecture 2.1 (as explained above). We prove this result in Section 5; however, we set up the necessary tools in the next section.

4 New properties of the compression operation

The proof of Theorem 3.10 will be based on the compression technique, which featured in the original proof of the classical Erdős-Ko-Rado Theorem [13].

For a non-empty set X and $x, y \in X$, let $\delta_{x,y}: 2^X \rightarrow 2^X$ be defined by

$$\delta_{x,y}(A) = \begin{cases} (A \setminus \{y\}) \cup \{x\} & \text{if } y \in A \text{ and } x \notin A; \\ A & \text{otherwise,} \end{cases}$$

and let $\Delta_{x,y}: 2^{2^X} \rightarrow 2^{2^X}$ be the *compression operation* (see [13]) defined by

$$\Delta_{x,y}(\mathcal{A}) = \{\delta_{x,y}(A) : A \in \mathcal{A}, \delta_{x,y}(A) \notin \mathcal{A}\} \cup \{A \in \mathcal{A} : \delta_{x,y}(A) \in \mathcal{A}\}.$$

Note that $|\Delta_{x,y}(\mathcal{A})| = |\mathcal{A}|$. It is well-known, and easy to check, that $\Delta_{x,y}(\mathcal{A})$ is intersecting if \mathcal{A} is intersecting; [14] provides a survey on the properties and uses of compression (also called *shifting*) operations in extremal set theory. We now establish new properties of compressions for the purpose of proving Theorem 3.10.

For any family \mathcal{A} , let \mathcal{A}^* denote the sub-family of \mathcal{A} consisting of those sets in \mathcal{A} that intersect each set in \mathcal{A} (i.e. $\mathcal{A}^* = \{A \in \mathcal{A} : A \cap B \neq \emptyset \text{ for any } B \in \mathcal{A}\}$), and let $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}^*$. So \mathcal{A}' consists of those sets in \mathcal{A} that do not intersect all the sets in \mathcal{A} , and \mathcal{A}^* is an intersecting family.

Lemma 4.1 *Let \mathcal{A} be a sub-family of $2^{[n]}$, and let $\mathcal{B} = \Delta_{i,j}(\mathcal{A})$ for some $i, j \in [n]$, $i \neq j$. Then:*

- (i) *if $A \in \mathcal{A}^*$ then $\delta_{i,j}(A) \in \mathcal{B}^*$;*
- (ii) *if $A \in \mathcal{A}^* \setminus \mathcal{B}^*$ then $\delta_{i,j}(A) \notin \mathcal{A}^*$;*
- (iii) *if $B \in \mathcal{B}^*$ then $\delta_{i,j}(B) \in \mathcal{B}^*$;*
- (iv) $|\mathcal{A}^*| \leq |\mathcal{B}^*|$.

Proof. The lemma is obvious if $\mathcal{A}^* = \emptyset$, so suppose $\mathcal{A}^* \neq \emptyset$. Fix $A \in \mathcal{A}^*$.

Obviously $\delta_{i,j}(A) \in \mathcal{B}$. Suppose $\delta_{i,j}(A) \notin \mathcal{B}^*$. Then $\delta_{i,j}(A) \cap C = \emptyset$ for some $C \in \mathcal{B}$. By definition of \mathcal{B} , $\delta_{i,j}(C)$ is also in \mathcal{B} , and hence both C and $\delta_{i,j}(C)$ are in \mathcal{A} . So A intersects both C and $\delta_{i,j}(C)$. From $\delta_{i,j}(A) \cap C = \emptyset$ and $A \cap C \neq \emptyset$ we get $i \notin C$, $\delta_{i,j}(A) \neq A$ (so $i \notin A$), $A \cap C = \{j\}$. But this yields the contradiction that $A \cap \delta_{i,j}(C) = \emptyset$. Hence (i).

Suppose $A \notin \mathcal{B}^*$. Assume that $\delta_{i,j}(A) \in \mathcal{A}^*$. Then $A \in \mathcal{B}$ (as both A and $\delta_{i,j}(A)$ are in \mathcal{A}) and $A \cap D = \emptyset$ for some $D \in \mathcal{B}$ (as $A \notin \mathcal{B}^*$). Since A intersects each set in \mathcal{A} , we must have $D = \delta_{i,j}(E) \neq E$ for some $E \in \mathcal{A}$, $A \cap E = \{j\}$ and $i \notin A \cup E$. But then $\delta_{i,j}(A) \cap E = \emptyset$, contradicting $\delta_{i,j}(A) \in \mathcal{A}^*$. So $\delta_{i,j}(A) \notin \mathcal{A}^*$. Hence (ii).

Suppose $B \in \mathcal{B}^*$. If $\delta_{i,j}(B) = B$ then obviously $\delta_{i,j}(B) \in \mathcal{B}^*$. Suppose $\delta_{i,j}(B) \neq B$. Then $B, \delta_{i,j}(B) \in \mathcal{A}$. Thus, since B intersects every set in \mathcal{B} and $i \notin B$, B intersects every set in \mathcal{A} , and hence $B \in \mathcal{A}^*$. By (i), $\delta_{i,j}(B) \in \mathcal{B}^*$. Hence (iii).

By (i), we can define a function $f: \mathcal{A}^* \rightarrow \mathcal{B}^*$ by

$$f(A) = \begin{cases} A & \text{if } A \in \mathcal{A}^* \cap \mathcal{B}^*; \\ \delta_{i,j}(A) & \text{if } A \in \mathcal{A}^* \setminus \mathcal{B}^*. \end{cases}$$

Suppose $A_1, A_2 \in \mathcal{A}^*$ such that $f(A_1) = f(A_2)$. Suppose $A_1 \in \mathcal{A}^* \cap \mathcal{B}^*$ and $A_2 \in \mathcal{A}^* \setminus \mathcal{B}^*$; then we have $\delta_{i,j}(A_2) = f(A_2) = f(A_1) = A_1 \in \mathcal{A}^*$, which is a contradiction because $\delta_{i,j}(A_2) \notin \mathcal{A}^*$ by (ii). Similarly, we cannot have $A_2 \in \mathcal{A}^* \cap \mathcal{B}^*$ and $A_1 \in \mathcal{A}^* \setminus \mathcal{B}^*$. If $A_1, A_2 \in \mathcal{A}^* \cap \mathcal{B}^*$ then we have $A_1 = f(A_1) = f(A_2) = A_2$. Finally, suppose $A_1, A_2 \in \mathcal{A}^* \setminus \mathcal{B}^*$. Then we have $\delta_{i,j}(A_1) = f(A_1) = f(A_2) = \delta_{i,j}(A_2)$ and, by (ii), $\delta_{i,j}(A_1) \neq A_1$ and $\delta_{i,j}(A_2) \neq A_2$. So $A_1 = \delta_{j,i}(\delta_{i,j}(A_1)) = \delta_{j,i}(\delta_{i,j}(A_2)) = A_2$. Therefore, no two distinct sets in \mathcal{A}^* are mapped by f to the same set in \mathcal{B}^* (i.e. f is injective). Hence (iv).

5 Proof of Theorem 3.10

We now prove Theorem 3.10. We adopt the strategy introduced in [3, 4] and also adopted in [5, 6, 8], which mainly is to determine, for the family \mathcal{F} under consideration, the largest real number $c \leq l/|\mathcal{F}|$ such that $|\mathcal{A}^*| + c|\mathcal{A}'| \leq l$ for any sub-family \mathcal{A} of \mathcal{F} , where l is the size of a largest intersecting sub-family of \mathcal{F} , and \mathcal{A}^* and \mathcal{A}' are as defined in Section 4; see [8] for a detailed general explanation.

Theorem 5.1 *Let \mathcal{H} be a hereditary sub-family of $2^{[n]}$ that is compressed with respect to an element x of $[n]$, and let \mathcal{A} be a sub-family of \mathcal{H} . Then*

$$|\mathcal{A}^*| + \frac{1}{n+1}|\mathcal{A}'| \leq |\mathcal{H}\langle x \rangle|,$$

and if $\mathcal{A}' \neq \emptyset$, then equality holds if and only if $\mathcal{A} = \mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$.

Proof. Since \mathcal{H} is compressed with respect to x , we have $x \in U(\mathcal{H})$ and hence $\mathcal{H}\langle x \rangle \neq \emptyset$. The result is trivial if $n = 1$, so we consider $n \geq 2$ and proceed by induction on n .

We may assume that $x = 1$. Let $\mathcal{L} = \mathcal{H}\langle 1 \rangle$. Let $\mathcal{B} = \Delta_{1,n}(\mathcal{A})$. Given that \mathcal{H} is compressed with respect to 1, we have $\mathcal{B} \subseteq \mathcal{H}$. Define

$$\begin{aligned} \mathcal{B}_1 &= \{B \in \mathcal{B} : n \in B\}, \\ \mathcal{B}_2 &= \{B \setminus \{n\} : B \in \mathcal{B}_1\}, \\ \mathcal{B}_3 &= \mathcal{B} \setminus \mathcal{B}_1 = \{B \in \mathcal{B} : n \notin B\}. \end{aligned}$$

Define $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ similarly. So $\mathcal{B}_2, \mathcal{L}_2 \subseteq \mathcal{H}_2 \subseteq 2^{[n-1]}$ and $\mathcal{B}_3, \mathcal{L}_3 \subseteq \mathcal{H}_3 \subseteq 2^{[n-1]}$. Also note that the properties of \mathcal{H} are inherited by \mathcal{H}_3 , that is, \mathcal{H}_3 is hereditary and compressed with respect to 1; the same holds for \mathcal{H}_2 unless $U(\mathcal{H}_2) = \emptyset$ (in which case \mathcal{H}_1 is either \emptyset or $\{\{n\}\}$). Define

$$\begin{aligned} \mathcal{C}_1 &= \{C \in \mathcal{B}_1 : 1 \in C, B \cap C = \{n\} \text{ for some } B \in \mathcal{B}_1\}, \\ \mathcal{C}_2 &= \{C \setminus \{n\} : C \in \mathcal{C}_1, C \setminus \{n\} \notin \mathcal{B}_3\}, \\ \mathcal{D} &= \mathcal{B}_2 \setminus \mathcal{C}_2, \\ \mathcal{E} &= \mathcal{B}_3 \cup \mathcal{C}_2. \end{aligned}$$

Obviously $\mathcal{C}_2 \subseteq \mathcal{B}_2$ and $\mathcal{D} \subseteq \mathcal{H}_2$. Given that \mathcal{H} is hereditary, we clearly have $\mathcal{C}_2 \subseteq \mathcal{H}_3$; so $\mathcal{E} \subseteq \mathcal{H}_3$. Note that $\mathcal{L}_2 = \mathcal{H}_2 \langle 1 \rangle$ and $\mathcal{L}_3 = \mathcal{H}_3 \langle 1 \rangle$. Therefore, by the inductive hypothesis, we have $|\mathcal{E}^*| + \frac{1}{n}|\mathcal{E}'| \leq |\mathcal{L}_3|$, and if $U(\mathcal{H}_2) \neq \emptyset$, then $|\mathcal{D}^*| + \frac{1}{n}|\mathcal{D}'| \leq |\mathcal{L}_2|$.

By definition of \mathcal{C}_2 , we have $\mathcal{B}_3 \cap \mathcal{C}_2 = \emptyset$ and hence $|\mathcal{E}| = |\mathcal{B}_3| + |\mathcal{C}_2|$. Since $\mathcal{C}_2 \subseteq \mathcal{B}_2$, we have $|\mathcal{D}| = |\mathcal{B}_2| - |\mathcal{C}_2|$. So $|\mathcal{D}| + |\mathcal{E}| = |\mathcal{B}_2| + |\mathcal{B}_3|$ and hence, since $|\mathcal{D}| + |\mathcal{E}| = |\mathcal{D}^*| + |\mathcal{D}'| + |\mathcal{E}^*| + |\mathcal{E}'|$ and $|\mathcal{B}_2| + |\mathcal{B}_3| = |\mathcal{B}| = |\mathcal{B}^*| + |\mathcal{B}'|$,

$$|\mathcal{D}^*| + |\mathcal{E}^*| + |\mathcal{D}'| + |\mathcal{E}'| = |\mathcal{B}^*| + |\mathcal{B}'|. \quad (1)$$

We now come to our main step, which is to show that $|\mathcal{B}^*| \leq |\mathcal{D}^*| + |\mathcal{E}^*|$. So suppose \mathcal{B}^* contains a set B .

First, suppose $n \notin B$. Then clearly B intersects all sets in $\mathcal{B}_2 \cup \mathcal{B}_3$ and hence, since $\mathcal{C}_2 \subseteq \mathcal{B}_2$, we have $B \in \mathcal{E}^*$. Also, $B \notin \mathcal{C}_2$ since $B \in \mathcal{B}_3$. In brief, we have

$$n \notin B \in \mathcal{B}^* \Rightarrow B \in (\mathcal{E}^* \setminus \mathcal{C}_2) \cap \mathcal{B}^*. \quad (2)$$

Now suppose $n \in B$, that is, $B \in \mathcal{B}_1$. Let $B^- = B \setminus \{n\}$. Clearly B^- intersects all sets in \mathcal{B}_3 . If $B^- \in \mathcal{C}_2$ then, since all sets in \mathcal{C}_2 contain 1, B^- also intersects each set in \mathcal{C}_2 , meaning that $B^- \in \mathcal{E}^*$. In brief, we have

$$n \in B \in \mathcal{B}^*, B \setminus \{n\} \in \mathcal{C}_2 \Rightarrow B \setminus \{n\} \in \mathcal{E}^* \cap \mathcal{C}_2. \quad (3)$$

Suppose $B^- \notin \mathcal{C}_2$. Then $B^- \in \mathcal{D}$. So suppose $B^- \notin \mathcal{D}^*$. Then $B^- \cap D = \emptyset$ for some $D \in \mathcal{D}$ and hence, setting $D^+ = D \cup \{n\}$, we have $B \cap D^+ = \{n\}$ and $D^+ \in \mathcal{B}_1$. Since $B \cap D = \emptyset$, D cannot be in \mathcal{B}_3 . Thus we must have $1 \notin D^+$ because otherwise we get $D^+ \in \mathcal{C}_1$ and hence $D \in \mathcal{C}_2$ (contradicting $D \in \mathcal{D}$). It follows that we must also have $1 \in B$ because otherwise we get $\delta_{1,n}(B) \cap D^+ = \emptyset$, contradicting Lemma 4.1(iii). So $B \in \mathcal{C}_1$. Thus, since $B^- \in \mathcal{D}$ implies $B^- \notin \mathcal{C}_2$, B^- must be in \mathcal{B}_3 and hence in \mathcal{B} . Since $B \cap D^+ = \{n\}$, we have $B^- \cap D^+ = \emptyset$ and hence $B^- \notin \mathcal{B}^*$. However, since B^- intersects all sets in \mathcal{B}_3 and $1 \in B^- \cap C$ for any $C \in \mathcal{C}_2$, we have $B^- \in \mathcal{E}^*$. So we have just shown that

$$n \in B \in \mathcal{B}^*, B \setminus \{n\} \notin \mathcal{C}_2, B \setminus \{n\} \notin \mathcal{D}^* \Rightarrow B \setminus \{n\} \in \mathcal{E}^* \setminus (\mathcal{C}_2 \cup \mathcal{B}^*). \quad (4)$$

Define

$$\begin{aligned} \mathcal{F}_1 &= \{F \in \mathcal{B}^* : n \notin F\}, \\ \mathcal{F}_2 &= \{F \in \mathcal{B}^* : n \in F, F \setminus \{n\} \in \mathcal{C}_2\}, \\ \mathcal{F}_3 &= \{F \in \mathcal{B}^* : n \in F, F \setminus \{n\} \notin \mathcal{C}_2, F \setminus \{n\} \notin \mathcal{D}^*\}, \\ \mathcal{F}_4 &= \{F \in \mathcal{B}^* : n \in F, F \setminus \{n\} \notin \mathcal{C}_2, F \setminus \{n\} \in \mathcal{D}^*\}. \end{aligned}$$

Clearly $|\mathcal{B}^*| = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_4|$ and $|\mathcal{F}_4| \leq |\mathcal{D}^*|$. Also, by (2) - (4), we have $|\mathcal{F}_1| \leq |(\mathcal{E}^* \setminus \mathcal{C}_2) \cap \mathcal{B}^*|$, $|\mathcal{F}_2| \leq |\mathcal{E}^* \cap \mathcal{C}_2|$ and $|\mathcal{F}_3| \leq |\mathcal{E}^* \setminus (\mathcal{C}_2 \cup \mathcal{B}^*)|$. Thus, since $(\mathcal{E}^* \setminus \mathcal{C}_2) \cap \mathcal{B}^*$, $\mathcal{E}^* \cap \mathcal{C}_2$ and $\mathcal{E}^* \setminus (\mathcal{C}_2 \cup \mathcal{B}^*)$ are disjoint sub-families of \mathcal{E}^* , we obtain $|\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \leq |\mathcal{E}^*|$. So $|\mathcal{B}^*| \leq |\mathcal{D}^*| + |\mathcal{E}^*|$ as required.

We now know that $|\mathcal{D}^*| + |\mathcal{E}^*| = |\mathcal{B}^*| + p$ for some integer $p \geq 0$. By (1), we therefore have $|\mathcal{D}'| + |\mathcal{E}'| = (|\mathcal{B}^*| + |\mathcal{B}'|) - (|\mathcal{B}^*| + p) = |\mathcal{B}'| - p$.

At this point, we need to divide the problem into two cases.

Case 1: $U(\mathcal{H}_2) \neq \emptyset$. So $|\mathcal{D}^*| + \frac{1}{n}|\mathcal{D}'| \leq |\mathcal{L}_2|$. Since we earlier obtained $|\mathcal{E}^*| + \frac{1}{n}|\mathcal{E}'| \leq |\mathcal{L}_3|$,

$$|\mathcal{D}^*| + |\mathcal{E}^*| + \frac{1}{n}(|\mathcal{D}'| + |\mathcal{E}'|) \leq |\mathcal{L}_2| + |\mathcal{L}_3| = |\mathcal{L}_1| + |\mathcal{L}_3| = |\mathcal{L}|.$$

We now have

$$|\mathcal{B}^*| + \frac{1}{n}|\mathcal{B}'| \leq |\mathcal{B}^*| + p + \frac{1}{n}(|\mathcal{B}'| - p) = |\mathcal{D}^*| + |\mathcal{E}^*| + \frac{1}{n}(|\mathcal{D}'| + |\mathcal{E}'|) \leq |\mathcal{L}|.$$

Since $|\mathcal{A}^*| + |\mathcal{A}'| = |\mathcal{A}| = |\mathcal{B}| = |\mathcal{B}^*| + |\mathcal{B}'|$, Lemma 4.1(iv) gives us $|\mathcal{A}^*| + \frac{1}{n}|\mathcal{A}'| \leq |\mathcal{B}^*| + \frac{1}{n}|\mathcal{B}'|$. So $|\mathcal{A}^*| + \frac{1}{n+1}|\mathcal{A}'| \leq |\mathcal{L}|$, and the inequality is strict if $\mathcal{A}' \neq \emptyset$.

Case 2: $U(\mathcal{H}_2) = \emptyset$. So \mathcal{H}_1 is either \emptyset or $\{\{n\}\}$. If $\mathcal{H}_1 = \emptyset$, then $\mathcal{H} \subseteq 2^{[n-1]}$ and hence the result follows by inductive hypothesis. Now suppose $\mathcal{H}_1 = \{\{n\}\}$. Then, since $\mathcal{B}_1 \subseteq \mathcal{H}_1$, we have $\mathcal{C}_1 = \mathcal{C}_2 = \emptyset$, which gives $\mathcal{D} = \mathcal{B}_2 \subseteq \{\emptyset\}$ and $\mathcal{E} = \mathcal{B}_3$. If $\mathcal{D} = \emptyset$ then the argument in Case 1 gives us the result.

Suppose $\mathcal{D} = \{\emptyset\}$. Since $\mathcal{D} = \mathcal{B}_2$, we have $\mathcal{B}_1 = \{\{n\}\}$ and hence $\{n\} \in \mathcal{B}$. By definition of \mathcal{B} , $\{1\}$ is also in \mathcal{B} . Therefore $\mathcal{B} \neq \mathcal{B}^*$; moreover, since there is no set in $\mathcal{B} \setminus \{\{n\}\}$ intersecting $\{n\}$, $\mathcal{B} = \mathcal{B}'$. Now consider \mathcal{H}_3 . From $\{1\} \in \mathcal{B} \subseteq \mathcal{H}$ we get $\{1\} \in \mathcal{H}_3$ and hence $\mathcal{H}_3 \neq \{\emptyset\}$. Since \mathcal{H} is hereditary, we have $\{\emptyset\} \in \mathcal{H}_3$, meaning that $\mathcal{H}_3^* = \emptyset$ and $\mathcal{H}_3 = \mathcal{H}_3'$. It follows by the inductive hypothesis that $\frac{1}{n}|\mathcal{H}_3| \leq |\mathcal{L}_3|$ (and hence $n|\mathcal{L}_3| - |\mathcal{H}_3| \geq 0$) and that equality holds only if $\mathcal{H}_3 = \{\emptyset\} \cup \binom{[n-1]}{1}$. So we have

$$\begin{aligned} |\mathcal{L}_3| - \left(|\mathcal{B}^*| + \frac{1}{n+1}|\mathcal{B}'| \right) &= |\mathcal{L}_3| - \frac{1}{n+1}|\mathcal{B}| = |\mathcal{L}_3| - \frac{1}{n+1}(|\mathcal{B}_1| + |\mathcal{B}_3|) \\ &\geq |\mathcal{L}_3| - \frac{1}{n+1}(1 + |\mathcal{H}_3|) = \frac{1}{n+1}((n+1)|\mathcal{L}_3| - 1 - |\mathcal{H}_3|) \\ &= \frac{1}{n+1}(n|\mathcal{L}_3| - |\mathcal{H}_3| + |\mathcal{L}_3| - 1) \geq \frac{1}{n+1}(|\mathcal{L}_3| - 1) \geq 0, \end{aligned} \quad (5)$$

where the last inequality follows from the fact that $\{1\} \in \mathcal{B} \subseteq \mathcal{H}$ and hence $\{1\} \in \mathcal{L}_3$. So $|\mathcal{B}^*| + \frac{1}{n+1}|\mathcal{B}'| \leq |\mathcal{L}|$. As in Case 1, we have $|\mathcal{A}^*| + \frac{1}{n+1}|\mathcal{A}'| \leq |\mathcal{L}|$ by Lemma 4.1. Suppose equality holds. Then $|\mathcal{B}^*| + \frac{1}{n+1}|\mathcal{B}'| = |\mathcal{L}|$. By the calculation in (5), we must therefore have $\frac{1}{n}|\mathcal{H}_3| = |\mathcal{L}_3|$, implying that $\mathcal{H}_3 = \{\emptyset\} \cup \binom{[n-1]}{1}$, and also $|\mathcal{B}_3| = |\mathcal{H}_3|$, implying that $\mathcal{B}_3 = \mathcal{H}_3$. Since $\mathcal{B}_1 = \mathcal{H}_1 = \{\{n\}\}$, $\mathcal{B} = \mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$. It clearly follows that $\mathcal{A} = \mathcal{B}$.

Finally, if $\mathcal{A} = \mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$, then $\mathcal{A}^* = \emptyset$, $\mathcal{A}' = \mathcal{A}$, $\mathcal{L} = \{\{1\}\}$, and hence $|\mathcal{A}^*| + \frac{1}{n+1}|\mathcal{A}'| = 1 = |\mathcal{L}|$. \square

Now for any non-empty family \mathcal{F} , let $l(\mathcal{F})$ be the size of a largest intersecting sub-family of \mathcal{F} , and let $\beta(\mathcal{F})$ be the largest real number $c \leq l(\mathcal{F})/|\mathcal{F}|$ such that $|\mathcal{A}^*| + c|\mathcal{A}'| \leq l(\mathcal{F})$ for any sub-family \mathcal{A} of \mathcal{F} .

Proof of Theorem 3.10. For any intersecting family $\mathcal{A} \neq \{\emptyset\}$, $\mathcal{A}^* = \mathcal{A}$ and $\mathcal{A}' = \emptyset$.

Thus, by Theorem 5.1, $\mathcal{H}\langle x \rangle$ is a largest intersecting sub-family of \mathcal{H} and hence $l(\mathcal{H}) = |\mathcal{S}|$. Also by Theorem 5.1, $\beta(\mathcal{H}) \geq \frac{1}{n+1}$. So we have $k \geq n+1 \geq \frac{1}{\beta(\mathcal{H})}$.

As in the proof of Theorem 3.7, let $\mathcal{A} = \bigcup_{i=1}^k \mathcal{A}_i$; so $\mathcal{A}^* = \bigcup_{i=1}^k \mathcal{A}_i^*$, $\mathcal{A}' = \bigcup_{i=1}^k \mathcal{A}_i'$ and $|\mathcal{A}'| = \sum_{i=1}^k |\mathcal{A}_i'|$. So we have

$$\sum_{i=1}^k |\mathcal{A}_i| = \sum_{i=1}^k |\mathcal{A}_i'| + \sum_{i=1}^k |\mathcal{A}_i^*| \leq |\mathcal{A}'| + k|\mathcal{A}^*| \leq k(|\mathcal{A}^*| + \beta(\mathcal{H})|\mathcal{A}'|) \leq kl(\mathcal{H}) = k|\mathcal{S}| \quad (6)$$

and hence, by Lemma 3.4,

$$\prod_{i=1}^k |\mathcal{A}_i| \leq \left(\frac{1}{k} \sum_{i=1}^k |\mathcal{A}_i| \right)^k \leq |\mathcal{S}|^k. \quad (7)$$

We now prove (a) and (b). It is trivial that the conditions in (a) and (b) are sufficient, so it remains to prove that they are also necessary.

Consider first $k > n+1$. Then $k > \frac{1}{\beta(\mathcal{H})}$. From (6) we see that $\sum_{i=1}^k |\mathcal{A}_i| = k|\mathcal{S}|$ only if $|\mathcal{A}'| = 0$ and $|\mathcal{A}_1^*| = \dots = |\mathcal{A}_k^*| = |\mathcal{A}^*| = |\mathcal{S}|$. So $\sum_{i=1}^k |\mathcal{A}_i| = k|\mathcal{S}|$ only if \mathcal{A} is a largest intersecting sub-family of \mathcal{H} and $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{A}$. It follows from (7) that $\prod_{i=1}^k |\mathcal{A}_i| = |\mathcal{S}|^k$ only if \mathcal{A} is a largest intersecting sub-family of \mathcal{H} and $\mathcal{A}_1 = \dots = \mathcal{A}_k = \mathcal{A}$.

Now consider $k = n+1$. If we still have $k > \frac{1}{\beta(\mathcal{H})}$, then we arrive at the same conclusion as in the previous case $k > n+1$. So suppose $k = \frac{1}{\beta(\mathcal{H})}$. Then $\beta(\mathcal{H}) = \frac{1}{n+1}$.

Suppose $\mathcal{H} \neq \{\emptyset\} \cup \binom{[n]}{1}$. Let $d = \frac{l(\mathcal{H})}{|\mathcal{H}|}$. Since $x \in U(\mathcal{H})$, $\mathcal{S} \neq \emptyset$. Thus, since $|\mathcal{H}| = |\{\emptyset\} \cup \bigcup_{i=1}^n \mathcal{H}\langle i \rangle|$, we get $|\mathcal{H}| \leq 1 + n|\mathcal{S}|$, and equality holds only if $|\mathcal{H}\langle i \rangle| = |\mathcal{S}|$ for all $i \in [n]$. So $|\mathcal{H}| \leq (n+1)|\mathcal{S}|$, and equality holds only if $|\mathcal{S}| = 1$ and $|\mathcal{H}\langle i \rangle| = |\mathcal{S}|$ for all $i \in [n]$. If $i \in [n]$ and $A \in \mathcal{H}\langle i \rangle$, then, since \mathcal{H} is hereditary, all subsets of A containing i are also in $\mathcal{H}\langle i \rangle$; thus, if $|\mathcal{H}\langle i \rangle| = 1$, then $\mathcal{H}\langle i \rangle$ must be $\{i\}$. Therefore, if $|\mathcal{H}| = (n+1)|\mathcal{S}|$, then $\mathcal{H}\langle i \rangle = \{i\}$ for all $i \in [n]$, but this gives the contradiction that $\mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$. So $|\mathcal{H}| < (n+1)|\mathcal{S}|$ and hence $\frac{1}{n+1} < \frac{|\mathcal{S}|}{|\mathcal{H}|} = d$. Now let $\mathcal{A} \subseteq \mathcal{H}$. If $\mathcal{A}' = \emptyset$, then obviously $|\mathcal{A}^*| + d|\mathcal{A}'| \leq l(\mathcal{H})$. If $\mathcal{A}' \neq \emptyset$, then $|\mathcal{A}^*| + \frac{1}{n+1}|\mathcal{A}'| < l(\mathcal{H})$ by Theorem 5.1. Thus, if c is the largest real number such that $c \leq d$ and $|\mathcal{A}^*| + c|\mathcal{A}'| \leq l(\mathcal{H})$ for any $\mathcal{A} \subseteq \mathcal{H}$, then $c > \frac{1}{n+1}$, which is a contradiction since $\beta(\mathcal{H}) = \frac{1}{n+1}$.

We have therefore shown that \mathcal{H} must consist of the sets $\emptyset, \{1\}, \{2\}, \dots, \{n\}$. It follows by the cross-intersection condition that we have the following:

- If one of the families $\mathcal{A}_1, \dots, \mathcal{A}_k$ consists of only one set A and $A \neq \emptyset$, then each of the others either consists of A only or is empty.
- If one of the families $\mathcal{A}_1, \dots, \mathcal{A}_k$ either has more than one set or has the set \emptyset , then the others must be empty.

These have the following immediate implications:

- If $\sum_{i=1}^k |\mathcal{A}_i| = k|\mathcal{S}|$, then $\sum_{i=1}^k |\mathcal{A}_i| = n+1$ (since $\mathcal{S} = \{\{x\}\}$ and $k = n+1$) and hence either $\mathcal{A}_1 = \dots = \mathcal{A}_k = \{\{y\}\}$ for some $y \in [n]$, or for some $i \in [n]$, $\mathcal{A}_i = \mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$ and $\mathcal{A}_j = \emptyset$ for each $j \in [n] \setminus \{i\}$.
- If $\prod_{i=1}^k |\mathcal{A}_i| = |\mathcal{S}|^k$, then $\sum_{i=1}^k |\mathcal{A}_i| = 1$ and hence $\mathcal{A}_1 = \dots = \mathcal{A}_k = \{\{y\}\}$ for some

$y \in [n]$.

Note that for any $y \in [n]$, $\{\{y\}\}$ is a largest intersecting sub-family of $\mathcal{H} = \{\emptyset\} \cup \binom{[n]}{1}$. \square

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